# Phase-equivalent potentials from supersymmetric quantum mechanics

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**Abstract.** Using *supersymmetry transformations we have constructed closed form* expressions for the phase-equivalent partners (PEP) of the Morse and Hulthén potentials and discussed some of their characteristic features in terms of a model calculation. We have found that (i) there is no Efimov effect for the PEP of the Morse potential and this results from the effect of elimination of the bound state of the parent potential, and (ii) the PEP of the Hulthén potential may have some interesting application in electron–ion scattering.

While mourning his unfortunate demise we dedicate this article to the memory of the late Kalyan Roy of Chandidas Mahavidyalaya, Burdwan, who worked with our group for more than three years.

## 1. Introduction

About 50 years ago Bargmann [1] discovered a manifold of potentials $V(r)$ which have the same spectral density for positive energies as a given potential $V_0(r)$. Each member of the manifold $V(r)$ is phase-equivalent to $V_0(r)$ but may differ in number of bound states [2, 3]. Bargmann's discovery has initiated more general studies in the so-called inverse scattering problem (ISP) with a view to establishing a causal connection underlying the relationship between scattering data and potentials. The inverse problem was attacked mainly by three groups [4-6] working more or less independently of each other. The solution found by Gel'fand and Levitan [4] appears to be more complete and mathematically more rigorous compared to the other two [5, 6]. The algorithms of the ISP can also be used to generate phase-equivalent potentials (PEP) [2].

In the recent past the Gel'fand–Levitan procedure has been revisited and re-examined by Sukumar [7] within the framework of supersymmetric quantum mechanics (SSQM) [8]. This fundamental observation has been followed by a number of important works. For example, Baye [9] has applied the SSQM to the determination of shallow nucleus–nucleus potentials which are phase-equivalent to a deep potential. Interestingly, *deep and shallow nature of nucleus–nucleus potential was a controversial* question for a long time [10]. Baye's method or equivalently that of Sukumar was applied by Amado [11] to the partial wave Coulomb problem to obtain a new potential which is phase-equivalent to the Coulomb potential. As opposed to the treatment of Baye, Amado's approach to the problem is analytic and helps one illuminate the role of SSQM to obtain a family of phase-equivalent potentials. More recently, Khare and

Sukhatme [12] have sought some generalization of the work of Amado to derive a parametric relationship among a number of previously studied potentials [3, 13, 14].

In this work we shall deal with (i) Morse and (ii) Hulthén potentials. Both these potentials have played a central role in nuclear, atomic and molecular physics. For example, the soft-core Morse potential can well account for $^3S_1$ and $^1S_0$ nucleon–nucleon scattering phaseshifts. The vibrations of a class of diatomic molecules can also be excellently described by the Morse function. On the other hand, the Hulthén potential serves as a model for the interaction between nucleons in deuteron. Also it can be regarded as a screened Coulomb potential. Here we work within the framework of ssQM, develop an analytical approach to construct phase-equivalent partners for the potentials in (i) and (ii) and try to gain some physical weight for the problem in the context of few-body dynamics by comparing appropriate numerical results. In section 2 we introduce the mathematical framework of the ssQM. Given any central potential, the results presented are directly applicable to calculate a phase-equivalent potential with one less bound state. These formulae can, however, be used iteratively to generate families of such potentials that have 2, 3 . . . less bound states. In section 3 we treat the Morse and Hulthén potentials to obtain results for PEPs with one less bound state. We recast all results in 'maximal' reduced form for straightforward numerical calculation. For illustrative purposes, we present plots of the PEPs together with the parent potentials. These plots, on the one hand, clarify the basic mechanism through which ssQM simulates elimination of bound states and generation of PEPs, and, on the other hand, provide an intuitive feeling for typical few-body phenomena [15] which depend only on features of the physical scattering or on-shell properties of the potential.

## 2. Supersymmetric transformations for PEPs

In supersymmetric quantum mechanics the $l$th partial wave radical Schrödinger Hamiltonian can be written in the factored form

$$H_0 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_0(r) = A_0^+ A_0^- + E_0^{(0)} \tag{1}$$

where $E_0^{(0)} < 0$ is the binding energy of the lowest bound state of $H_0$ such that

$$H_0 \psi_0^{(0)}(r) = E_0^{(0)} \psi_0^{(0)}(r). \tag{2}$$

Clearly, the energy $E_0^{(0)}$ is the factorization energy of the Schrödinger operator $H_0$ [16, 17]. The superscript (0) on $\psi$ stands for the ground state wavefunction while the subscript 0 merely indicates that the wavefunction $\psi_{0l}^{(0)}(r)$ belongs to $H_0$. The superscript and subscript on $E$ have similar meaning. We shall use analogous notations for the partner potentials. Here we work in units in which $\hbar = m = 1$. The potential $V_0(r)$ includes the centrifugal barrier and is given by

$$V_0(r) = \frac{l(l+1)}{2r^2} + U_0(r). \tag{3}$$

The potential $V_0(r)$ is assumed to be regular in the usual sense [18]. The operators $A_0^\pm$ are given by

$$A_0^\pm = \frac{1}{\sqrt{2}}\left(\pm\frac{\mathrm{d}}{\mathrm{d}r} + W_0\right). \tag{4}$$

In terms of bound state wavefunction the 'superpotential' $W_0$ is written as

$$W_0 = \frac{d}{dr} \ln \psi_{0l}^{(0)}(E_0^{(0)}, r). \tag{5}$$

The Hamiltonian $H_0$ has a supersymmetric partner $H_1$ given by [7]

$$H_1 = A_0^- A_0^+ + E_0^{(0)} = H_0 + \frac{d}{dr} W_0. \tag{6}$$

It shares the same spectrum as $H_0$ except the ground state $E_0^{(0)}$ of $H_0$. The Hamiltonian $H_1$ corresponds to a potential

$$V_1(r) = \frac{(l+1)(l+2)}{2r^2} + U_0(r) - \frac{d^2}{dr^2} \ln \left( \frac{\psi_{0l}^{(0)}(E_0^{(0)}, r)}{r^{l+1}} \right). \tag{7}$$

In writing (7) we have used $\psi_{0l}^{(0)} \sim r^{l+1}$ for small $r$ and removed the singularity at $r = 0$. When supersymmetry is unbroken the eigenvalues and eigenstates of $H_1$ and $H_0$ obey the mapping

$$E_1^{(m)} = E_0^{(m+1)} \tag{8a}$$

and

$$\psi_{1l}^{(m)}(E_1^{(m)}, r) = (E_0^{(m+1)} - E_0^{(0)})^{-1/2} A_0^- \psi_{0l}^{(m+1)}(E_0^{(m+1)}, r) \qquad m = 0, 1, 2 \ldots . \tag{8b}$$

The relations in (8) refer to excited bound states. For positive energies $E = \frac{1}{2}k^2$ ($k$ = on-shell momentum) the eigenfunction relation of (8b) can be generalized [7, 19] to read

$$\psi_{1l}(E, r) = (E - E_0^{(0)})^{-1/2} A_0^- \psi_{0l}(E, r). \tag{9}$$

The result for $V_1(r)$ looks like a non-singular potential with angular momentum $l+1$. Since the supersymmetry relation is concerned only with the radial Schrödinger equation for some fixed value of $l$, the correct interpretation of (7) is that the potential $V_1(r)$ is singular at the origin [9], and this singularity arises from the term $(l+1) \times (l+2)/r^2$. From (3) and (7), it is clear that the potentials $V_0(r)$ and $V_1(r)$ have identical asymptotic behaviour. Despite this, they are not phase-equivalent. This can easily be seen by using the asymptotic form of (9) [7, 9]. To construct a PEP for $V_0(r)$ one proceeds by introducing a supersymmetric partner $H_2$ of $H_1$ with $E_0^{(0)}$ as separation/factorization energy. The energy $E_0^{(0)}$ lies below the ground state energy of $H_1$. The Hamiltonian $H_2$ can be generated in terms of regular solutions of the Schrödinger equation

$$H_1 \psi_{1l}(E_0^{(0)}, r) = E_0^{(0)} \psi_{1l}(E_0^{(0)}, r). \tag{10}$$

The wavefunction $\psi_{1l}(E_0^{(0)}, r)$ is not square integrable, but can be chosen regular at $r = 0$ as follows.

A particular solution of (10) is $[\psi_{0l}^{(0)}(E_0^{(0)}, r)]^{-1}$ for which we can write the superpotential as

$$\hat{W} = \frac{d}{dr} \ln [\psi_{0l}^{(0)}(E_0^{(0)}, r)]^{-1}. \tag{11}$$

The Riccati form of (10) is given by

$$W^2 + W' = 2(V_1(r) - E_0^{(0)}) \tag{12}$$

where

$$W = \frac{d}{dr} \ln \psi_{1l}(E_0^{(0)}, r). \tag{13}$$

To construct a regular solution of (10), we shall make use of the solution of (12). A general solution of (12) can be written in the form

$$W = \hat{W} + \phi. \tag{14}$$

From (12) and (14) we have

$$\phi' + 2\phi \hat{W} + \phi^2 = 0. \tag{15}$$

In (12) and (15) prime denotes differentiation with respect to $r$. The Bernoulli-type equation in (15) can be reduced to a linear one by using the transformation

$$\phi = y^{-1} \tag{16}$$

to get

$$y' + 2\hat{W}y = 1. \tag{17}$$

From (16) and (17)

$$\phi = \frac{d}{dr} \ln\left[ \int_0^r \psi_{0l}^{(0)^2}(E_0^{(0)}, r') \, dr' \right]. \tag{18}$$

In writing (18), we have focused our attention on those PEPs which have one less bound state and disregarded other possibilities considered by Khare and Sukhatme [12]. Combining (11), (14) and (18) we have

$$\psi_{1l}(E_0^{(0)}, r) = \frac{1}{\psi_{0l}^{(0)}(E_0^{(0)}, r)} \int_0^r \psi_{0l}^{(0)^2}(E_0^{(0)}, r') \, dr' \tag{19}$$

the desired regular solution of (10). In terms of the result in (19), the phase-equivalent supersymmetric partner $H_2$ of the Hamiltonian $H_1$ can be written as

$$H_2 = A_1^- A_1^+ + E_0^{(0)} = -\frac{1}{2}\frac{d^2}{dr^2} + V_2(r) \tag{20}$$

with

$$A_1^{\pm} = -A_0^{\mp} + \frac{1}{\sqrt{2}}\frac{d}{dr} \ln\left[ \int_0^r \psi_{0l}^{(0)^2}(E_0^{(0)}, r') \, dr' \right]. \tag{21}$$

From (20) and (21), the PEP $V_2(r)$ of $V_0(r)$ is

$$V_2(r) = V_0(r) - \frac{d^2}{dr^2}\left[ \int_0^r \psi_{0l}^{(0)^2}(E_0^{(0)}, r') \, dr' \right]. \tag{22}$$

The potentials $V_1(r)$ and $V_2(r)$ share exactly the same bound state spectrum. This implies that $V_2(r)$ has all the same bound states as $V_0(r)$ except for the lowest. Thus

for any given energy $E$, the solutions of the Hamiltonians $H_2$, $H_1$ and $H_0$ are related by

$$\psi_{2l}(E, r) = \begin{cases} (E - E_0^{(0)})^{-1/2} A_1^- \psi_{1l}(E, r) & (23a) \\ (E - E_0^{(0)})^{-1} A_1^- A_0^- \psi_{0l}(E, r). & (23b) \end{cases}$$

From (4), (5), (21) and (23b), we have

$$\psi_{2l}(E, r) = \psi_{0l}(E, r) - \frac{\psi_{1l}(E, r)}{(E - E_0^{(0)})} \left( \frac{\mathrm{d}}{\mathrm{d}r} \ln \int_0^r \psi_{0l}^{(0)2}(E_0^{(0)}, r') \, \mathrm{d}r' \right)$$

$$\times \left( \frac{\mathrm{d}}{\mathrm{d}r} \ln \frac{\psi_{0l}^{(0)}(E_0^{(0)}, r)}{\psi_{0l}(E, r)} \right). \tag{24}$$

Since $\psi_{0l}^{(0)}(E_0^{(0)}, r)$ is square integrable, the function $\psi_{2l}(\cdot, \cdot) \to \psi_{0l}(\cdot, \cdot)$ as $r \to \infty$. This demonstrates the phase-equivalence of $V_2(r)$ and $V_0(r)$.

## 3. PEPs for the Morse and Hulthén potentials

Some useful applications of the results presented in section 2 are now in order. With this in mind, we apply them to construct phase-equivalent potentials corresponding to the Morse and Hulthén potentials. For both these potentials the Schrödinger equation can be solved analytically only for $l = 0$. We shall, therefore, be interested in the $s$-wave case and present all results in analytic form. The inevitable numerical routine will be invoked only at a latter stage of the game. Henceforth we shall omit the subscript $l = 0$.

### 3.1. Morse potential

The Morse potential is given by

$$V_0^M(r) = D \, \mathrm{e}^{2(r_1-r)/d} - 2D \, \mathrm{e}^{(r_1-r)/d}. \tag{25}$$

It has a minimum $-D$ at $r = r_1$. The parameter $d$ is a measure of the range of the potential. It supports a finite number of bound states (say $N$) with energy eigenvalues

$$E_0^{M(m)} = -\frac{\alpha_1^2}{2d^2} + \left( m + \frac{1}{2} \right) \frac{\alpha_1}{d^2} - \frac{1}{2d^2} \left( m + \frac{1}{2} \right)^2 \qquad m = 0, 1, 2, \ldots, N-1. \tag{26}$$

In (26) $\alpha_1 = d\sqrt{2D}$ and $N$ is the largest integer less than $(\alpha_1 + \frac{1}{2})$. The unnormalized ground state eigenfunction of $V_0^M(r)$ is

$$\psi_0^{M(0)}(E_0^{M(0)}, r) = \mathrm{e}^{-(\alpha_1 - \frac{1}{2})r/d} \exp[-\alpha_1 \, \mathrm{e}^{(r_1-r)/d}] \tag{27}$$

with energy eigenvalue

$$E_0^{M(0)} = -\frac{1}{2} \left( \frac{\alpha_1 - \frac{1}{2}}{d} \right)^2. \tag{28}$$

If we take $E_0^{M(0)}$ as the factorization energy of $H_0^M$, the partner potential $V_1^M(r)$ of $V_0^M(r)$ can be obtained in the form

$$V_1^M(r) = V_0^M(r) + \frac{\alpha_1}{d^2} \exp\left( \frac{r_1 - r}{d} \right). \tag{29}$$

In writing (29), we have made use of

$$A_1^{M\pm} = \frac{1}{\sqrt{2}}\left(\pm\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\alpha_1 - \frac{1}{2}}{d} + \frac{\alpha_1}{d}\exp\left(\frac{r_1 - r}{d}\right)\right). \tag{30}$$

To construct the expression for $V_2^M(r)$ we have first obtained the regular solution of

$$H_1^M \psi_1^M(E_0^{M(0)}, r) = E_0^{M(0)}\psi_1^M(E_0^{M(0)}, r) \tag{31}$$

in the form

$$\psi_1^M(E_0^{M(0)}, r)$$

$$= \sum_{n=0}^{\infty} \frac{(-2\alpha_1)^n\, \mathrm{e}^{nr_1/d}}{n!}\, r \exp[(\alpha_1 - \tfrac{1}{2})r/d]$$

$$\times \exp[\alpha_1\, \mathrm{e}^{(r_1-r)/d}]\, {}_1F_1\left(1; 2; -\frac{2\alpha_1 + n - 1}{d}r\right) \tag{32}$$

where ${}_1F_1(\,\cdot\,)$ stands for the regular confluent hypergeometric function. In terms of the wavefunction in (32) the operators $A_1^{M\pm}$ corresponding to those in (21) are given by

$$A_1^{M\pm} = \frac{1}{\sqrt{2}}\left(\pm\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\mathrm{d}}{\mathrm{d}r}\ln\psi_1^M(E_0^{M(0)}, r)\right). \tag{33}$$

From (22) and (33) we find the result for $V_2^M$ in the form

$$V_2^M(r) = V_0^M(r) - Z^M(r)\left[\frac{2\alpha_1}{d}\mathrm{e}^{(r_1-r)/d} - 2\left(\frac{\alpha_1 - \frac{1}{2}}{d}\right) - Z^M(r)\right] \tag{34}$$

where

$$Z^M(r) = \frac{\exp\left[-2\left(\alpha_1 - \frac{1}{2}\right)\dfrac{r}{d}\right]\exp[-2\alpha_1\,\mathrm{e}^{(r_1-r)/d}]}{\displaystyle\sum_{n=0}^{\infty}\frac{(-2\alpha_1)^n\,\mathrm{e}^{nr_1/d}}{n!}\, r\, {}_1F_1\left(1; 2; -\dfrac{2\alpha_1 + n - 1}{d}r\right)}. \tag{35}$$

For the values of the parameters $D = 109.61$ MeV, $r_1 = 0.8531$ fm and $d = 0.3584$ fm the potential in (25) can reproduce the phaseshifts for ${}^3S_1$ nucleon–nucleon scattering up to a laboratory energy of 300 MeV and supports only one bound state at 2.218 MeV [20]. This is the deuteron binding energy. Compared to the depth of the potential this bound state energy is very small. Thus, if this potential is used as a pairwise interaction to study the three-body bound states, one expects to come across the so-called Efimov effect [15]. It is important to note that the Efimov effect is a characteristic of the on-shell properties of the potential and is independent of its details. Reasonably, one would like to know whether this typical three-body effect will also be observed when we try to employ the phase-equivalent potential $V_2(r)$ in place of $V_0(r)$. This question has recently been beautifully expounded by Amado *et al* [21]. We shall shortly come to this point. Meanwhile, we present in figure 1 the plots of the reduced potentials corresponding to $V_0^M(r)$, $V_1^M(r)$ and $V_2^M(r)$ as a function of $r$. We represent the variation of $(m/\hbar^2)V_0^M(r)$ and its phase-equivalent partner $(m/\hbar^2)V_2^M(r)$ by solid lines. We have used a dashed line to show similar variation of the auxiliary potential $(m/\hbar^2)V_1^M(r)$. The auxiliary potential just misses the ground state and becomes coincident with $(m/\hbar^2)V_0^M(r)$ at and after 3.2 fm. Inspite of this, $(m/\hbar^2)V_0^M(r)$ and $(m/\hbar^2)V_1^M(r)$ are phase inequivalent. The phase-equivalent potential $(m/\hbar^2)V_2^M(r)$
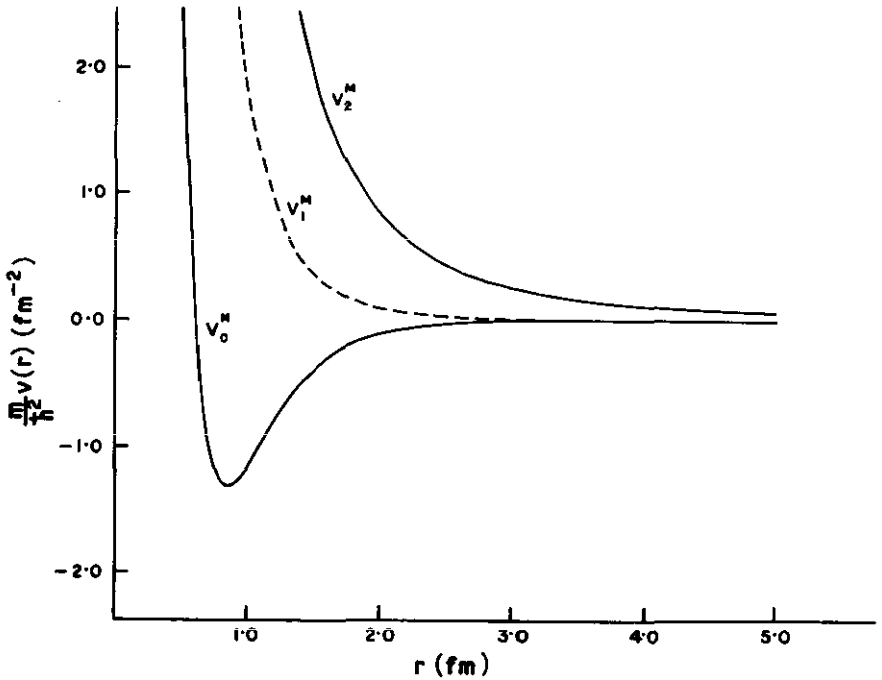
**Figure 1.** The Morse potential and its phase-equivalent partner as a function of $r$ in units of $\hbar^2/m(=82.94 \text{ MeV fm}^2)$, $m$ being the nucleon mass.

is far more singular at small $r$ and heals less rapidly to $(m/\hbar^2)V_0^M(r)$. This represents the typical characteristics of Pursey-type potentials [12, 13]. The singularity structure of the phase-equivalent potential can explicitly be shown by careful analysis of (7) and (22) and plays a role in the studies of scattering with Pauli-blocked bound states in compound systems [22].

Equation (24) demonstrates phase-equivalence of $V_0(r)$ and $V_2(r)$ in the wavefunction level. The phase-equivalence, however, holds good even in the Green function level [21]. In the present context we have verified the following.

(i) The interacting outgoing wave Green function $G_0^{(+)M}(k, r, r')$ and $G_2^{(+)M}(k, r, r')$ for Morse potential and its phase-equivalent partner are asymptotically equal.

(ii) The Green function $G_0^{(+)M}(k, r, r')$ tends to trigger the Efimov divergence when used for calculating the three-body integral kernel of a bound state problem.

(iii) No Efimov effect is observed for the function $G_2^{(+)M}(k, r, r')$ of the phase-equivalent potential.

The observations in (ii) and (iii) constitute a direct verification for the caution to be followed in embedding the two-body dynamics in a many-body system [21].

### 3.2. Hulthén potential

The two-parameter Hulthén potential ($V_0$ and $a$) is given by

$$V_0^H(r) = -V_0 \frac{e^{-r/a}}{(1-e^{-r/a})}. \tag{36}$$

In the limit $a \to \infty$, the potential in (36) goes over to the Coulomb potential if $V_0 a^2 = e^2 = 1$ (in atomic units). As noted earlier the Schrödinger equation for the

Hulthén potential can be solved only for $l = 0$. The ground state wavefunction and energy eigenvalue for the Hulthén potential is given by

$$\psi_0^{H(0)}(r) = a \exp\left[-\left(\frac{\beta^2-1}{2a}\right)r\right](1-e^{-r/a}) \tag{37}$$

$$E_0^{H(0)} = -\frac{1}{2}\left(\frac{\beta^2-1}{2a}\right)^2 \tag{38}$$

with

$$\beta^2 = 2V_0 a^2 > 0. \tag{39}$$

Because of (39), (36) can be written as

$$V_0^H(r) = -\frac{\beta^2}{2a^2}\frac{e^{-r/a}}{(1-e^{-r/a})}. \tag{40}$$

We shall use this form of $V_0^H(r)$ in all our subsequent discussion. The auxiliary potential corresponding to $V_0^H(r)$ is given by

$$V_1^H(r) = V_0^H(r) + \frac{e^{-r/a}}{a^2(1-e^{-r/a})^2}. \tag{41}$$

The regular solution of interest for $V_1^H(r)$ is obtained as

$$\psi_1^H(E_0^{H(0)}, r) = \frac{a^3}{3}(1-e^{-r/a})^2 \exp\left[-\left(\frac{\beta^2-1}{2a}\right)r\right]$$
$$\times {}_2F_1\left(3, 1-2a\left(\frac{\beta^2-1}{2a}\right); 4; 1-e^{-r/a}\right). \tag{42}$$

Using the wavefunction in (42) we have obtained the PEP, $V_2^H(r)$, of $V_0^H(r)$ in the form

$$V_2^H(r) = V_0^H(r) - Z^H(r)\left[-2\left(\frac{\beta^2-1}{2a}\right) + \frac{2e^{-r/a}}{a(1-e^{-r/a})} - Z^H(r)\right] \tag{43}$$

where

$$Z^H(r) = \frac{3}{a}\frac{e^{-2((\beta^2-1)/2a)r}}{(1-e^{-r/a})\,{}_2F_1(3, 1-2a((\beta^2-1)/2a); 4; 1-e^{-r/a})}. \tag{44}$$

In (44) ${}_2F_1(\cdot)$ stands for the Gaussian hypergeometric function.

The phase-equivalent partner $V_2^C(r)$ for the Coulomb potential $V_0^C(r)$ can be written in terms of incomplete gamma function $\gamma(a, x)$ [12]. The function $\gamma(a, x)$ is related to the confluent hypergeometric function $({}_1F_1(\cdot))$ by

$$\gamma(a, x) = a^{-1}x^a\,{}_1F_1(a; a+1; -x). \tag{45}$$

Taking notice of this and using the limits $a \to \infty$ and $(\beta^2-1)/2a \to 1$ together with the transformation formula

$$\lim_{b\to\infty} {}_2F_1\left(a, b; c; \frac{x}{b}\right) = {}_1F_1(a; c; x) \tag{46}$$

we have obtained the result for $V_2^C(r)$ from our expression for $V_2^H(r)$. Admittedly, our derived result refers only to the s-wave case.

In figure 2 we portray the variation of $V_0^H(r)$, $V_1^H(r)$ and $V_2^H(r)$ as a function of $r$ for $a = 50$ au. These curves are drawn by solid lines. In this figure we display the
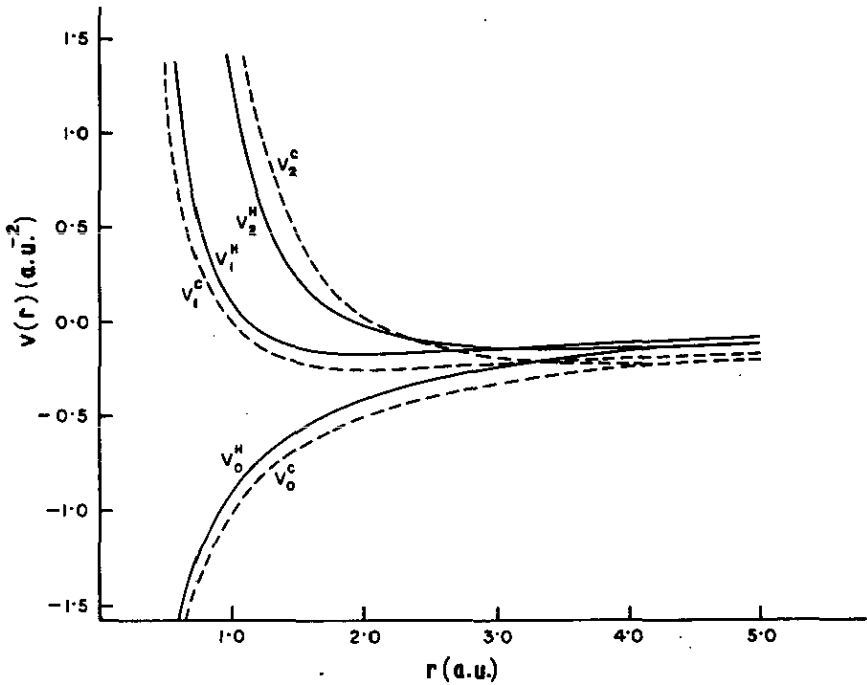
**Figure 2.** The Hulthén potential and its phase-equivalent partner as a function of $r$ in au.

corresponding results for the Coulomb potential by dashed curves. Looking closely into these curves we see that the $\mathrm{sgn}(V_0^C(r) - V_0^H(r)) = \mathrm{sgn}(V_1^C(r) - V_1^H(r))$ for all $r$ but $\mathrm{sgn}(V_2^C(r) - V_2^H(r)) = \mathrm{sgn}(V_0^C(r) - V_0^H(r))$ only after $r = 2.2$ au. For small $r$ region, $\mathrm{sgn}(V_2^C(r) - V_2^H(r)) = -\mathrm{sgn}(V_0^C(r) - V_0^H(r))$. We have checked that this property is repeated if we iterate the supersymmetric procedure to construct higher order auxiliary and phase-equivalent potentials. Thus the curves in figure 2 help one visualize the points of contrast and similarity between the Hulthén and Coulomb potentials when treated within the framework of SSQM. The result of this section may be of some interest to atomic physicists since the interaction between an electron and $Li^+$ ion is a screened Coulomb force with the 1S state occupied.

## Acknowledgments

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